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Turbulent boundary layer on a circular cylinder: the low-wavenumber surface pressure spectrum due to a low-Mach-number flow

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The effect of surface curvature on the low-wavenumber–frequency spectrum of turbulence-induced surface pressure fluctuations is considered and an estimate for the associated flow noise is obtained. The form of the low-wavenumber cross-spectral density of pressure on the surface of an infinitely long, rigid circular cylinder of radius a due to a statistically stationary turbulent boundary-layer flow at a low Mach number is determined. Viscous effects are ignored. It is shown that, in contrast to the case of an infinite plane surface, the pressure spectrum is finite everywhere in the wavenumber plane $(k, n/a)$ except for a logarithmic, integrable, singularity at the acoustic wavenumber corresponding to the axisymmetric mode $(n/a = 0)$; k and n/a being, respectively, the downstream and circumferential wavenumber. For non-axisymmetric modes $(|n/a| > 0)$, the spectrum has two finite peaks in the radiative domain $|k| < \omega/c$; ω being the frequency and c being the sound speed. For $\omega a/c$ large, the peaks occur in the vicinity of the total acoustic surface wavenumbers $\kappa = \pm\omega/c$ $(\kappa = (k^2+n^2/a^2)^{\frac{1}{2}})$ and the principal contribution which determines the peak characteristics can be identified as being due to creeping rays emanating from turbulent sources on the cylinder out of the line of sight of the associated receiver point. For large value of $\omega a/c$, the point pressure spectrum and the associated radiated sound vary logarithmically with $(\omega a/c)^{\frac{2}{3}}$; corresponding estimates for cylinders of moderate and small radius are also obtained. For an almost plane cylinder, it is shown that the effect of curvature may be included by a suitable simple modification of the form of the pressure spectrum for an infinite plane surface.

1. Introduction

The pressure fluctuations induced by a turbulent boundary layer over a rigid surface have frequencies ω which are typically $O(U_c/\Delta)$, where Δ is the boundary-layer thickness and U_c is the speed with which the boundary-layer eddies convect. The fluctuations induce self-noise, cause structural vibrations and radiate sound. The cross-correlation of surface pressure is a measure of the intensity of these effects, so it is of interest to study its characteristics.

For flow over a plane surface at low Mach numbers $M = U_c/c$, where c is the speed of sound, a typical plot (for example, Chase 1980) of the cross-spectral density of wall pressure against streamwise wavenumber (figure 1a) features a broad peak at the convective streamwise wavenumber $O(\omega/U_c)$ together with a relatively narrower peak at the critical (lower) wavenumber $|\kappa| = \omega/c$, where κ is the total surface wavenumber $(k^2+k_3^2)^{\frac{1}{2}}$. Thus the two peaks are fairly well apart and it makes sense to consider the characteristics of the spectral density in the vicinity of the two peaks

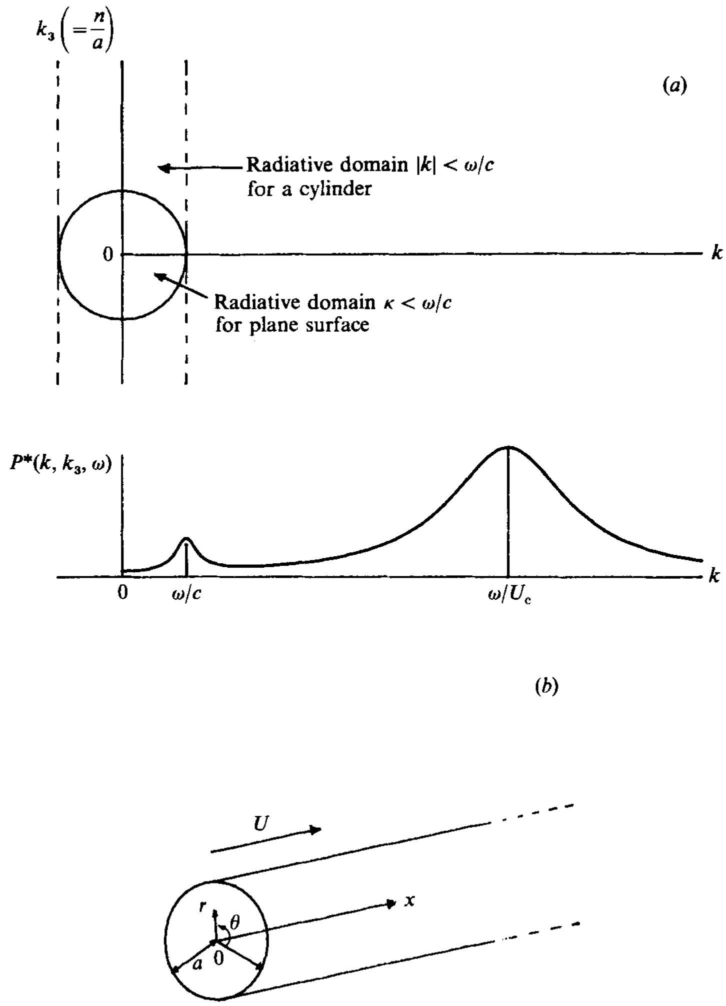


FIGURE 1. (a) Typical fixed-frequency cross-spectral density of pressure on a plane surface with $k_3 = 0$. The radiative domains in the wavenumber plane for the plane surface and for a circular cylinder are shown above. (b) The coordinate system.

separately. The characteristics of the pressure spectrum in the vicinity of the convective peak are principally governed by incompressible mechanics, while the low-wavenumber characteristics for $|\kappa| < \omega/c$ are, as noted by Ffowcs Williams (1965), governed by the effects of compressibility. Here, we shall be concerned with the latter characteristics of the spectral density. For a fixed frequency, the lengthscales associated with low wavenumber considerations are large compared with those associated with viscous diffusion, so the fluid may be regarded as inviscid.

If the plane surface is of infinite extent, then the spectral density of surface pressure has a non-integrable singularity at the acoustic wavenumber, the spectral density being proportional to the response function $(\omega^2/c^2)\,|\kappa^2-\omega^2/c^2|^{-1}$. Thus for a given non-zero frequency, the contribution to the correlation area from the low-wavenumber domain of the wavenumber plane is infinite! Apparently, in the Lighthill (1952) formulation, no allowance is made for the weak interaction of the sound field of a turbulent source element with the turbulence through which it propagates, so that it does not decay fast enough with distance for the integrated contribution from distant acoustic sources in the infinite plane to be finite. Although damping of the sound field due to turbulence is a slow process, over a distance $O(M^{-2}c/\omega)$, it is significant enough to make the integrated contribution bounded (Crow 1967); the effect of viscous surface shear also acts over a comparable distance (Howe 1979). At low Mach numbers, the distances involved are rather large and if the turbulent boundary layer extends over a region with typical dimension $L \ll M^{-2}c/\omega$, the damping can be ignored and the neglect of interaction of sound with turbulence in Lighthill's formulation is justified. For a finite value of L (Ffowcs Williams 1965, 1982; Bergeron 1973), the singularity at the acoustic wavenumber is integrable. Away from the acoustic wavenumber, $|\kappa^2 - \omega^2/c^2| \gg \omega/cL$, the spectrum coincides with that for the infinite plane. The coefficient of the point pressure spectrum is proportional to $\ln(\omega L/c)$. (See also recent work of Howe 1987.)

Here we consider a turbulent boundary-layer flow over a smooth infinitely long rigid cylinder of radius a, and determine the associated cross-spectral density of surface pressure using the Lighthill formulation. Our principal aim is to investigate how the cylindrical geometry may modify the nature of the power spectrum in the vicinity of the acoustic wavenumber from the known form for a plane rigid surface in the absence of any acoustic damping by turbulence or viscosity. The effect of having a non-rigid surface, which, as shown by Dowling (1983), can be significant in underwater applications, is therefore excluded here. The effects of boundary-layer growth and mean shear are also ignored; the former is expected to smear to a small extent the bandwidths associated with pertinent frequencies, while the available evidence for a plane surface (Dowling 1983) suggests that for low-Mach-number underwater flows the latter effect is small. Further, the neglect of viscous effects implies that the no-slip condition at the surface of the cylinder must be abandoned; however, as in the case of the plane surface, the slip velocity here does not contribute to the spectral density. (The author is grateful to Professor J. E. Ffowcs Williams for drawing his attention to this point.) The turbulent boundary layer is assumed to be statistically stationary, transient motions based on initial conditions having decayed away.

In §2, Lighthill's (1952) equation for small density fluctuations is presented and used to derive an expression for the pressure in terms of the unknown nonlinear source terms associated with Reynolds stress.

In §3, an expression for the cross-power spectral density of surface pressure is obtained in terms of a product of a response function associated with the cylindrical geometry and a term essentially involving the source functions. The spectrum for a plane surface is obtained from this expression in the appropriate limit of letting the radius of the cylinder become infinite. The result is in agreement with that obtained by Bergeron.

In §4, the form of the spectrum in the low wavenumber domain is considered. For a cylinder, the radiative domain corresponds to the strip $|k| \leqslant \omega/c$, where k is the streamwise wavenumber (see figure 1a). When considering low wavenumbers, the term involving the source functions is expected to be well behaved (see Bergeron 1973) and the nature of the spectrum is principally governed by the response function. If n/a denotes the azimuthal wavenumber, then for $n/a = 0$ the response function has a logarithmic, integrable, singularity at the acoustic wavenumber. For $n/a \neq 0$, the response function has finite peaks at $\kappa = \pm\kappa_{\mathrm{m}}$ where $(\omega^2/c^2) + (n^2/a^2) > \kappa_{\mathrm{m}}^2 > \omega^2/c^2$; the peaks become broader and lower with increasing values of n/a. Away from the peaks, the response function is well behaved and decays to zero as $|\kappa| \rightarrow \infty$. Thus, again in contrast to the case of the infinite plane surface, the low-wavenumber contribution to the point spectrum is finite, and is estimated in §4.

Three cases are considered: (*a*) a small cylinder, $\Delta \ll a \ll c/\omega$; (*b*) a cylinder of radius comparable with the acoustic wavelength; and (*c*) a large cylinder, $a \gg c/\omega$. In case (*a*), the principal contribution to the point spectrum has a coefficient of $O(\Delta/a)^4$. For $n/a = 0$, the pressure spectrum is in agreement with the leading-order approximation obtained by Chase & Noiseux (1982) for axisymmetric flow, although the extra terms in the expression for the pressure spectrum, suggested by these authors as being concomitant to the slip velocity at the surface, in fact do not arise.

In case (*b*), the point spectrum is estimated by assuming that the term involving the source function can be approximated by its value at a fixed low wavenumber and numerically integrating with respect to k the summation of the response function over n.

In case (*c*) it is shown that, for large a, the coefficient of the peak value of the pressure spectrum is proportional to $M^4(\omega a/U_c)^2 n^{-\frac{4}{3}}$ and the width of the peak is $O(n^{\frac{2}{3}}/a)$. Further away from the peaks, the spectrum coincides with that for an infinite plane surface. This suggests that for an almost plane cylinder, the effect of finite curvature may be allowed for by approximating the response function by

$$\frac{\omega^2}{c^2}\left(\left|\kappa^2(1-\beta^2)-\frac{\omega^2}{c^2}\right|+\kappa^2\beta^2\right)^{-1},$$

where β is $O((\omega a/c)^{-\frac{1}{3}})$ and varies with the wave-vector angle. Then, for non-zero values of n/a, the response function has a peak value β^{-2} and peak width $O(\beta)$. For $n/a = 0$, the approximate response function is singular at the acoustic streamwise wavenumber; however, except in the case of axisymmetric flow (considered here for the purpose of comparison), the contribution to the point spectrum is finite with a coefficient proportional to $\ln((\omega a/c)^{\frac{2}{3}})$. This contribution corresponds to that from equivalent turbulent sources distributed over a finite plane disk of radius $L = O(a^{\frac{2}{3}}(\omega/c)^{-\frac{1}{3}})$. For this value of L, L^{-1} is the exponential decay rate of creeping rays on a cylinder (Jones 1979). Thus the point pressure spectrum is finite, the dominant low-wavenumber contribution to the point spectrum being due to creeping rays emanating from turbulent sources within a distance L (measured along the surface of the cylinder) of the point. It may be noted that the contribution implied by ray theory from sources in line of sight of the point corresponds to that from equivalent sources distributed over a plane disk of smaller radius $L = O(a\Delta)^{\frac{1}{2}}$, $\omega\Delta/c$ being very small for low-Mach-number flows. Hence, in approximating the low-wavenumber contribution to the point spectrum or sound intensity it is insufficient to consider individual contributions merely from turbulent sources directly in line of sight of the point; the creeping-ray contribution from sources out of the line of sight also needs to be considered, it being the dominant contribution.

It has recently come to light that Howe (1987) has, independently, considered the effect of general surface curvature (in both streamwise and transverse directions) on the singularity at the acoustic wavenumber. Having identified the dominance of the creeping-ray contribution to the spectral peak at the acoustic wavenumber, he evaluates the leading-order effect of general wall curvature. One consequence of having a general curvature is that the logarithmic singularity corresponding to axisymmetric spectral elements in the case of a cylinder, referred to above, is removed and the spectrum is finite everywhere. He suggests an approximate response function, which for a cylinder is essentially similar to the above; the main difference is that for $k_3 > 0$, he estimates the peak to lie at the acoustic wavenumber instead of at its true position just off the acoustic position. Consequently for $k_3 < \omega/c$, he slightly underestimates the height of the peak.

The approximate response function for an almost plane cylinder cannot be used in the case of axisymmetric flow; this is considered separately in Appendix B. It is shown that the contribution to the point spectrum is again finite but with a coefficient proportional to $\ln(\omega a/c)$. These features are discussed in §5 where, following Bergeron, the estimates for the intensity of radiated sound for turbulent flow past a cylinder are obtained.

2. Basic theory

We consider turbulent boundary-layer flow over an infinitely long, smooth and rigid circular cylinder of radius a. The flow velocity away from the cylinder surface is uniform and parallel to the axis of the cylinder. The fluid is regarded as inviscid, viscous effects being negligible in the present low-wavenumber acoustic considerations. This implies that we must allow for a slip velocity at the cylinder surface, requiring only the normal component of velocity to vanish at the surface. We approximate the mean speed in the turbulent boundary layer by a constant U_c, which is of the same order as the speed in the outer stream.

Further, we assume a linearized equation of state,

$$p - p_0 = c^2(\rho - \rho_0), \tag{2.1}$$

where $p(\boldsymbol{x}, t)$ and $\rho(\boldsymbol{x}, t)$ respectively denote fluid pressure and density and p_0, ρ_0 and c are constants, c being the speed of acoustic waves. The linear approximation is adequate for considerations of flow at low Mach numbers.

Then, in a frame of reference fixed with respect to the stream, the pressure field is governed by Lighthill's equation (Dowling, Ffowcs Williams & Goldstein 1978),

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p = \frac{\partial^2}{\partial x_i\,\partial x_j}(\rho_0 u_i u_j) \quad (i,j = 1,2,3), \tag{2.2}$$

where the summation over Cartesian components is implied, u_i denoting the ith component of fluctuating fluid velocity with zero mean value. The density ρ in the term on the right-hand side has been replaced by its ambient value ρ_0, consistent with the approximation (2.1); both approximations imply errors of $O(M^2)$ in (2.2). The equations tacitly assume that the effect of the sound field on turbulence is negligible for flow at low Mach numbers (cf. Crow 1970).

In terms of cylindrical coordinates x, r, θ, where Ox is along the length of the axis of the cylinder (figure 1*b*), on denoting the fluid velocity components u_x, u_r, u_θ, (2.2) may be written

$$\mathrm{L}p(x,r,\theta,t) = Q(x,r,\theta,t) - \frac{1}{r}\frac{\partial}{\partial r}T_{\theta\theta}, \tag{2.3}$$

where
$$\mathrm{L} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial x^2},$$

$$Q = \frac{\partial^2}{\partial x_i\,\partial x_j}T_{ij} + \frac{2}{r}\frac{\partial^2}{\partial x_i\,\partial r}(rT_{ri}) + \frac{1}{r}\frac{\partial^2}{\partial r^2}(rT_{rr}) \quad (i,j = 1,3)$$

and where, without ambiguity,

$$T_{xx} \equiv T_{11} = \rho_0 u_x^2, \quad T_{xr} \equiv T_{12} = \rho_0 u_x u_r, \quad T_{x\theta} \equiv T_{13} = \rho_0 u_x u_\theta, \quad \text{etc.},$$

with
$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial x_3} = \frac{1}{r}\frac{\partial}{\partial \theta}.$$

The momentum equations imply that $\partial p/\partial r = T_{\theta\theta}/r$ at $r = a$. Further, we note that T_{ij} and its derivatives vanish as $r \rightarrow \infty$ so that

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p = 0 \quad \text{as } r \rightarrow \infty.$$

That is, p is a wave field as $r \rightarrow \infty$ and we require that it is outgoing in the limit. In general, the boundary conditions as $x \rightarrow \pm\infty$ within the boundary layer are lacking. Here we confine attention to the case where T_{ij} and its first and second derivatives also vanish in this limit, so that p again satisfies the wave equation in the limit. Hence, we require that p is an outgoing wave field everywhere at infinity. A solution to (2.3) for p may be obtained by introducing a Green function, G; $G(\boldsymbol{x}, t \,|\, \boldsymbol{x}_0, t_0)$ satisfies

$$\mathrm{L}G(\boldsymbol{x}, t \,|\, \boldsymbol{x}_0, t_0) = \frac{\delta(r - r_0)}{r}\delta(\theta - \theta_0)\,\delta(x - x_0)\,\delta(t - t_0), \tag{2.4}$$

together with the causality condition and the boundary conditions $\partial G/\partial r = 0$ at $r = a$, and G an outgoing wave at infinity. We proceed to solve for G by means of Fourier transforms, using an asterisk to denote the Fourier transform, defined for any function $A(x, ., \theta, t)$ as

$$A^*(., k, n, \omega) = \frac{1}{(2\pi)^3}\int\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} A(x, ., \theta, t)\,\mathrm{e}^{-\mathrm{i}n\theta - \mathrm{i}kx - \mathrm{i}\omega t} a\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t, \tag{2.5a}$$

so that
$$A(x, ., \theta, t) = \frac{1}{a}\int\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} A^*\,\mathrm{e}^{\mathrm{i}n\theta + \mathrm{i}kx + \mathrm{i}\omega t}\mathrm{d}k\,\mathrm{d}\omega. \tag{2.5b}$$

We take Fourier transforms with respect to x and t (and finite Fourier transform with respect to θ) of (2.4), writing $G^*(r, k, n, \omega \,|\, \boldsymbol{x}_0, t_0)$ for the transform of $G(\boldsymbol{x}, t \,|\, \boldsymbol{x}_0, t_0)$, to obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \gamma^2 - \frac{n^2}{r^2}\right)G^* = -\frac{\delta(r - r_0)}{r}E_0, \tag{2.6}$$

where
$$\gamma^2 = k^2 - \frac{\omega^2}{c^2}, \quad E_0 = \mathrm{e}^{-\mathrm{i}n\theta_0 - \mathrm{i}kx_0 - \mathrm{i}\omega t_0}. \tag{2.7}$$

A solution which satisfies the appropriate boundary conditions is given by

$$G^*(r, k, n, \omega \,|\, \boldsymbol{x}_0, t_0) = \frac{E_0}{K_n'(\gamma a)}\begin{cases} g(r_0, r), & r \leqslant r_0, \\ g(r, r_0), & r \geqslant r_0, \end{cases} \tag{2.8}$$

where
$$g(s, t) = K_n(\gamma s)\,[K_n'(\gamma a)\,I_n(\gamma t) - I_n'(\gamma a)\,K_n(\gamma t)]$$

and K_n, I_n are modified Bessel functions. For $\gamma^2 > 0$, G^* decays to zero exponentially as $r \rightarrow \infty$. For $\gamma^2 < 0$, G^* corresponds to an outgoing wave if that branch of γ is chosen for which its imaginary part is positive. in particular,

$$G^*(a, k, n, \omega \,|\, \boldsymbol{x}_0, t_0) = \frac{K_n(\gamma r_0)\,E_0}{\gamma a K_n'(\gamma a)}. \tag{2.9}$$

Then it can be shown that the pressure at a field point is given by

$$p(x,r,\theta,t) = \iint_{-\infty}^{\infty}\int_{-\pi}^{\pi}\int_{a}^{\infty}\left\{G(\boldsymbol{x},t\,|\,\boldsymbol{x}_0,t_0)\,Q(\boldsymbol{x}_0,t_0)+\frac{T_{\theta\theta}}{r_0}\frac{\partial}{\partial r_0}G\right\}r_0\,\mathrm{d}r_0\,\mathrm{d}\theta_0\,\mathrm{d}x_0\,\mathrm{d}t_0, \quad (2.10a)$$

the boundary conditions $v_r = 0$ and $\partial G/\partial r = 0$ at $r = a$, and the radiation condition at infinity ensuring that the surface integral vanishes. Transient motions based on initial conditions are ignored.

Since $T_{ij}(i,j = 1,2,3)$ and its first derivative vanish at infinity and $T_{ri}(i = 1,2,3)$ and $\partial T_{rr}/\partial r$ vanish at the surface of the cylinder, (2.10a) can be integrated by parts to give

$$p(x,r,\theta,t) = \iint_{-\infty}^{\infty}\int_{-\pi}^{\pi}\int_{a}^{\infty}\left[T_{ij}\frac{\partial^2 G}{\partial x_{0i}\,\partial x_{0j}}+2T_{rt}\frac{\partial^2 G}{\partial x_{0i}\,\partial r_0}+T_{rr}\frac{\partial^2 G}{\partial r_0^2}+\frac{T_{\theta\theta}}{r_0}\frac{\partial G}{\partial r_0}\right]r_0\,\mathrm{d}r_0\,\mathrm{d}\theta_0\,\mathrm{d}x_0\,\mathrm{d}t_0. \quad (2.10b)$$

Equation (2.10b) was given by Dowling *et al.* (1978) in terms of Cartesian tensors.

3. Cross-power spectral density

Assuming a statistically stationary boundary layer of thickness Δ, the cross-correlation of pressure on the surface of the cylinder is given by

$$P(x-\tilde{x},\theta-\tilde{\theta},t-\tilde{t}) = \langle p(x,a,\theta,t)\,p(\tilde{x},a,\tilde{\theta},\tilde{t})\rangle, \quad (3.1)$$

where $\langle\ \rangle$ denote an ensemble average. The cross-power spectral density of the surface pressure is given by the Fourier transform of P and is denoted here, in view of the convention defined by (2.5), by $P^*(k,n,\omega)$. We substitute for p in (3.1) from (2.10b), with G^* given by (2.9), and write $J^*_{ijlm}(r,\tilde{r},k,n,\omega)$ $(i,j,l,m = 1,2,3$, the suffix 2, without ambiguity, being synonymous with suffix 'r') for the Fourier transform of the source functions,

$$J_{ijlm}(x-\tilde{x},r,\tilde{r},\theta-\tilde{\theta},t-\tilde{t}) = \langle T_{ij}(\boldsymbol{x},t)\,T_{lm}(\tilde{\boldsymbol{x}},\tilde{t})\rangle. \quad (3.2)$$

Then on defining

$$\kappa^2 = k^2+\left(\frac{n}{a}\right)^2; \quad \theta_1 = \frac{k}{\kappa}; \quad \theta_3 = \frac{n}{\kappa a} \quad (\theta_1 = \theta_3 = 0 \quad \text{if } \kappa = 0), \quad (3.3)$$

$$\Gamma^2 = \gamma^2+\left(\frac{n}{a}\right)^2 = \kappa^2-\left(\frac{\omega}{c}\right)^2, \quad (3.4)$$

γ being as in (2.7), it can be shown that

$$P^*(k,n,\omega) = \rho_0^2\,U_c^3\,\Delta^5\,|aF(n,\gamma a)|^2\,\hat{R}(k,n,\omega), \quad (3.5)$$

with

$$F(n,z) = \frac{K_n(z)}{zK'_n(z)} \quad (3.6)$$

and

$$\hat{R}(k,n,\omega) = \hat{R}_0(k,n,\omega)+2\,\mathrm{Re}\left(\frac{\hat{R}_1(k,n,\kappa)}{aF(n,\gamma a)}\right)+\frac{\hat{R}_2(k,n,\omega)}{|aF(n,\gamma a)|^2}, \quad (3.7)$$

where

$$\hat{R}_0 = \kappa^4\theta_i\theta_j\theta_l\theta_m f_{ijlm} - 2\Gamma^2\kappa^2\theta_l\theta_m f_{22lm} + \Gamma^4 f_0 - \frac{4\kappa^2\theta_3^2 f_{2323}}{a^2} + \frac{4\kappa\theta_3(\kappa^2\theta_l\theta_m f_{23lm} - \Gamma^2 f_{3222})}{a},$$

$$\hat{R}_1 = 2\kappa\theta_i(\Gamma^2 g_{i222} + \kappa^2\theta_l\theta_m g_{i2lm}) + \frac{\Gamma^2 g_0}{a} - \frac{\kappa^2(4\theta_i\theta_3 g_{i223} + \theta_l\theta_m g_{0lm})}{a} - \frac{2\kappa\theta_3 g_1}{a},$$

$$\hat{R}_2 = 4\kappa\theta_i\left(\kappa\theta_l h_{i22l} - \frac{h_{0i}}{a}\right) + \frac{h_0}{a^2},$$

the indices taking the values $i, j, l, m = 1, 3$ and the summation convention being implied. $f[(k\Delta, n, \gamma a, (\omega\Delta/U_c), (\Delta/a)]$, $g[k\Delta, n, \gamma a, (\omega\Delta/U_c), (\Delta/a)$, and $h[k\Delta, n, \gamma a, (\omega\Delta/U_c), (\Delta/a)]$ are non-dimensionalized functions and are defined in Appendix A.

The details of the spectral form of the source functions J^*_{ijlm}, J^*_{rrrr}, etc. are not available. At low Mach numbers, M, we assume that they are well behaved in M and can be expanded in a Taylor series about $M = 0$. Thus, to leading order, the source functions are given by incompressible flow theory. By expanding the integrand in (A 4), it can be shown that the corresponding coefficients f, g, h are, to leading order, also given by incompressible flow theory. We further assume that at low streamwise wavenumbers, $k\Delta \ll 1$, these coefficients are well behaved in k and for sufficiently small values of k can be approximated by their values at $k = 0$; in fact, we use these values to estimate the size of the coefficients in the region $k\Delta \ll 1$. In the low-wavenumber ($k\Delta \ll 1$) approximation, we write

$$f\left(n, \frac{\omega\Delta}{U_c}\right) = f\left(0, n, 0, \frac{\omega\Delta}{U_c}, \frac{\Delta}{a}\right),$$

with similar expressions for g and h, so that in the approximation f, g, h are devoid of any structure in k and are unaffected by compressibility effects. We denote the low-wavenumber approximation to $\hat{R}(k, n, \omega)$ by

$$\hat{R}\left(k, n, \omega \,|\, f\left(n, \frac{\omega\Delta}{U_c}\right), g\left(n, \frac{\omega\Delta}{U_c}\right), h\left(n, \frac{\omega\Delta}{U_c}\right)\right) \equiv \left(\frac{\omega}{c}\right)^4 \hat{R}_\ell\left(\frac{kc}{\omega}, \frac{nc}{\omega a}, \frac{\omega\Delta}{U_c}\right), \tag{3.8}$$

where the explicit dependence on Δ/a is suppressed. We denote the corresponding low-wavenumber approximation to $P^*(k, n, \omega)$ by $P^*_\ell(k, n, \omega)$.

It may be noted that for large k, $|aF(n, \gamma a)|^2 \sim \kappa^{-2}$, so that from (3.5), and in view of (3.7),

$$P^*_\ell\,(k, n, \omega) = O(\rho_0^2 U_c^3 \Delta^5 \kappa^2). \tag{3.9}$$

Further,
$$\hat{R}(0, 0, \omega) = \left(\frac{\omega}{c}\right)^4 f_0 - 2\,\mathrm{Re}\left(\frac{\omega^2 g_0/c^2}{a^2 F(0, \mathrm{i}\omega a/c)}\right) + \frac{h_0/a^2}{|aF(0, \mathrm{i}\omega a/c)|^2} \tag{3.10}$$

and
$$P^*(0, 0, \omega) = \rho^2 U_c^3 \Delta^5 \left\{\left|aF\left(0, \mathrm{i}\frac{\omega a}{c}\right)\right|^2 \hat{R}(0, 0, \omega)\right\}. \tag{3.11}$$

For non-zero values of $\omega a/c$, $F(0, \mathrm{i}(\omega a/c))$ is finite so that $P^*(0, 0, \omega)$ is finite. This is consistent with the corresponding result for a plane surface (Ffowcs Williams 1965). However, if we let $c \to \infty$, so as to recover the incompressible case, we obtain

$$P^*(0, 0, \omega) \to \rho^2 U_c^3 \Delta^5 \frac{h_0}{a^2}. \tag{3.12}$$

Thus, the surface-pressure spectrum in the incompressible limit is finite. This is unlike the case of the plane surface, where $P^*(0,0,\omega)$ vanishes in the incompressible limit; this result is recovered from (3.12) by letting $a \to \infty$. We can infer from (3.12) that the surface pressure spectrum for incompressible flow is finite in κ at $\kappa = 0$. This result agrees with that obtained by J. E. Ffowcs Williams (personal communication) and Chase & Noiseux (1982) who considered the axisymmetric case for compressible and incompressible flow for $\gamma a \ll 1$. However, the extra contribution to the pressure spectrum, suggested by the latter authors as being concomitant to the slip velocity at the surface, do not arise if, as we have shown, the correct boundary condition for inviscid flow, namely $\partial p/\partial r = T_{\theta\theta}/r$ at $r = a$, is applied; Chase & Noiseux assume, incorrectly, that $\partial p/\partial r = 0$ at $r = a$.

3.1. *Plane-surface limit*

The power spectrum for an infinite plane surface is recovered from (3.5) by taking the limit $n \to \infty$, $a \to \infty$ but $n/a \to k_3$, a finite value. We note that in this limit

$$\frac{K_n(\gamma r)}{K_n(\gamma a)} \to \mathrm{e}^{-\Gamma y}; \quad aF \to -\frac{1}{\Gamma}, \tag{3.13}$$

where $y = r-a$ and where for $\Gamma^2 < 0$, that branch of Γ is chosen for which its imaginary part is positive. Then, in the limit, writing

$$\frac{n}{a} \equiv k_3 \tag{3.14}$$

and extending the definition of Fourier transform in (2.5) with respect to θ to the spanwise direction of the infinite plate, we have from (3.7),

$$\hat{R}_\infty(k, k_3, \omega) = \lim_{\substack{n\to\infty \\ n/a\to k_3}} \hat{R}(k, n, \omega),$$

$$\hat{R}_\infty(k, k_3, \omega) = \kappa^4\theta_i\theta_j\theta_l\theta_m f_{ijlm} - 2\kappa^2\theta_l(\Gamma^2\theta_m f_{22lm} - 2|\Gamma^2|\theta_i h_{i22l}) + \Gamma^4 f_0 - 4\theta_i\kappa\,\mathrm{Re}\,(\Gamma\bar{\Gamma}^2 g_{i22l} + \kappa^2\Gamma\theta_l\theta_m g_{i2lm}) \quad (i,j,l,m = 1,3), \tag{3.15}$$

where the plane-surface limit is implied in the definitions (A 3) of f, g and h. Thus, from (3.5), the cross-power spectral density for an infinite plane surface is

$$P^*_\infty(k, k_3, \omega) = \frac{\rho_0^2 U_c^3 \Delta^5}{|\Gamma|^2}\hat{R}_\infty(k, k_3, \omega). \tag{3.16}$$

In the low-Mach-number, low-wavenumber approximation, f, g and h, to leading order, are again assumed to be devoid of any structure in k and to be approximately unaffected by compressibility. We further assume that they are well behaved in k_3 and for sufficiently small values of k_3 can be approximated by their values at $k_3 = 0$. Thus in (A 3) we write

$$f\left(\frac{\omega\Delta}{U_c}\right) \approx \lim_{a\to\infty} f\left(0, 0, 0, \frac{\omega\Delta}{U_c}, \frac{\Delta}{a}\right)$$

etc., and denote the corresponding approximation to $\hat{R}_\infty(k, n, \omega)$ by

$$\hat{R}_\infty\left(k, k_3, \omega \,|\, f\left(\frac{\omega\Delta}{U_c}\right), g\left(\frac{\omega\Delta}{U_c}\right), h\left(\frac{\omega\Delta}{U_c}\right)\right) \equiv \left(\frac{\omega}{c}\right)^4 \hat{R}_{\infty\ell}\left(\frac{\omega k}{c}, \frac{\omega k_3}{c}, \frac{\omega\Delta}{U_c}\right), \tag{3.17a}$$

and the corresponding approximation to the pressure spectrum $P^*_\infty(k, k_3, \omega)$ by

$$P^*_{\infty\ell}(k,k_3,\omega) = \frac{\rho_0^2 U_c^3 \Delta}{|\Gamma|^2}\left(\frac{\omega\Delta}{c}\right)^4 \hat{R}_{\infty\ell}\left(\frac{\omega k}{c}, \frac{\omega k_3}{c}, \frac{\omega\Delta}{U_c}\right). \tag{3.17b}$$

Equation (3.17*b*) agrees with the corresponding expression given by Bergeron (1973). We note from (3.15) that $\hat{R}_\infty(k, k_3, \omega)$ is non-zero at the acoustic wavenumber for which $\Gamma = 0$, so that, in view of (3.17*b*), $P^*_{\infty\ell}(k, k_3, \omega)$ is singular there and the low-wavenumber contribution to the point pressure spectrum, given by

$$P^*_{\infty\ell}(\omega) = \iint_{\kappa\Delta \ll 1} P^*_{\infty\ell}(k, k_3, \omega)\,\mathrm{d}k_1\,\mathrm{d}k_3 \tag{3.18}$$

is infinite, the singularity being non-integrable. This feature is due to the infinite extent of the surface. The sound field of an acoustic source does not decay sufficiently fast with distance that the contribution to the point pressure spectrum from the acoustic field due to a plane infinite layer of turbulence is finite. Ffowcs Williams (1965, 1982) and Bergeron (1973) showed that for a plane surface of finite extent, an appropriately defined point pressure spectrum is finite and varies as the logarithm of the dimensions of the plane.

4. Spectrum in the radiative domain

We expect that the acoustic waves generated by the flow in the boundary layer have frequencies $\omega = O(U_c/\Delta)$. Thus for the low-Mach-number flow under consideration, $\omega\Delta/c \ll 1$.

The radiative domain in the wavenumber plane for the surface pressure spectrum associated with a circular cylinder extends over the infinite strip $|k| \leqslant \omega/c$, in contrast to that for a plane surface where it extends over the circle $\kappa \leqslant \omega/c$. However, from the evidence available for a plane surface, we expect that for $|n| \gg \omega a/c$, the pressure spectrum will decay as $(\omega a/cn)^2$. Thus the main contribution is still expected to come from $\kappa = O(\omega/c)$.

We note that for large values of n

$$|aF(n,\gamma a)|^2 \sim \frac{1}{\Gamma^2}, \quad |n| > |\gamma a|, \tag{4.1}$$

where $\Gamma(k, n)$ is given by (3.4), so that $|aF(n, \gamma a)|^2$ is $O(a^2/n^2)$. Since the pressure spectrum $P^*(k, n, \omega)$ is expected to be of this order in the azimuthal wavenumber, the factor $\hat{R}(k, n, \omega)$ in (3.3), to leading order, must be independent of n for large n. For $k \ll \Delta^{-1}$ and $|n| > a\Delta^{-1}$ we may, there estimate $\hat{R}(k, n, \omega)$ by $(\omega/c)^4 \hat{R}_\ell(kc/\omega, c/\omega\Delta, \omega\Delta/U_c)$, where $\hat{R}_\ell(kc/\omega, nc/\omega a, \omega\Delta/U_c)$ is defined in (3.8).

Thus for $k \ll \Delta^{-1}$, we have from (3.5) and (3.8) that the density of the surface pressure spectrum is approximately given by

$$P^*_\ell(k,n,\omega) = \rho_0^2 U_c^3 \left(\frac{\omega\Delta}{c}\right)^2 \Delta^3 \left|\frac{\omega a}{c} F(n,\gamma a)\right|^2 \begin{cases} \hat{R}_\ell\left(\dfrac{kc}{\omega}, \dfrac{nc}{\omega a}, \dfrac{\omega\Delta}{U_c}\right), & |n| \leqslant a\Delta^{-1}, \\[2ex] \hat{R}_\ell\left(\dfrac{kc}{\omega}, \dfrac{c}{\omega\Delta}, \dfrac{\omega\Delta}{U_c}\right), & |n| > a\Delta^{-1}. \end{cases} \tag{4.2}$$

The factor $\hat{R}_\ell$ is devoid of any singularities and if the source integrals $f(n, \omega\Delta/U_c)$, $g(n, \omega\Delta/U_c)$ etc. in (3.8) are all of the same order, we expect that it is a fairly well-behaved function of the wavenumbers for $k \ll \Delta^{-1}$. Thus the nature of the pressure spectrum there is principally determined by the factor $|(\omega a/c)\, F(n,\gamma a)|^2$.

In the radiative region, we may use the properties of the Bessel functions to express $|F(n,\gamma a)|^2$ in terms of Hankel functions,

$$|F(n,\gamma a)|^2 = \frac{|K_n(\gamma a)|^2}{|\gamma a K_n'(\gamma a)|^2} = \frac{|H_n^{(1)}(\psi a)|^2}{|\psi a H_n^{(1)\prime}(\psi a)|^2}, \tag{4.3}$$

where
$$\psi^2 = -\gamma^2 = \frac{\omega^2}{c^2} - k^2.$$

For a given value of $\omega a/c$, $|F(n,\gamma a)|^2$ is plotted against the streamwise wavenumber in figure 2(*a*) for various values of n. For $n = 0$, $|F(n,\gamma a)|^2$ is singular at $k = \pm\omega/c$, while for a non-zero value of n such that $|n| < N = O(\omega a/c)$, it has finite peaks at positions $k = \pm k_{\mathrm{m}}$, where $\omega^2/c^2 > k_{\mathrm{m}}^2 > \omega^2/c^2 - n^2/a^2$; in fact, for given n, k_{m} corresponds to the value of k for which $(\gamma a)^2 - n^2 = -n^2(1+|\gamma a F(n,\gamma a)|^2)^{-1}$ (we denote $k_{\mathrm{m}}^2 + n^2/a^2$ by κ_{m}^2). The peaks become broader and lower with increasing n. For $|n| > N$, $|F(n,\gamma a)|^2$ has a maximum value at $k = 0$ from which it decays according to (4.1).

In the vicinity of the acoustic streamwise wavenumbers, $k = \pm\omega/c(|\gamma a| \to 0)$,

$$\left|\frac{\omega a}{c} F(n,\gamma a)\right|^2 \sim \begin{cases} \dfrac{\omega^2 a^2}{c^2}\ln^2(|\gamma a|), & n = 0 \\ \dfrac{\omega^2 a^2}{n^2 c^2} & n \neq 0 \end{cases} \quad \text{as } |\gamma a| \to 0, \tag{4.4}$$

so that $|\omega a/c\, F(0,\gamma a)|^2$ is logarithmically singular at $k = \pm\omega/c$. From (3.7) and (3.8), for non-zero values of f_{1111}, $\hat{R}_\ell(k\omega/c,\ 0,\ \omega\Delta/c)$ is finite, so that $P_\ell^*(k,0,\omega)$ is also logarithmically singular at the acoustic wavenumber. However, unlike the case of the plane surface, the singularity is integrable. Further, for $n \neq 0$, $|(\omega a/c) F(n,0)|^2$ is finite and decays with increasing n as $(\omega a/nc)^2$. Thus the contribution $P_{\ell\mathrm{cr}}^*(\omega,\epsilon)$ to the point spectrum $P^*(\omega)$ from the acoustic region, $|\gamma| < \omega\epsilon/c$, $\epsilon \ll 1$, is finite and is approximately given by

$$P_{\ell\mathrm{cr}}^*(\omega,\epsilon) = \int_{|\gamma| < \omega\epsilon/c} \mathrm{d}k \left\{\frac{1}{a}\sum_n P_\ell^*(k,n,\omega)\right\}. \tag{4.5}$$

If we substitute the expression (4.2) for $P_\ell^*(k,n,\omega)$ into (4.5) and approximate the factor $\hat{R}_\ell$ in P_ℓ^*, for each n by its value at the acoustic wavenumber with $n = 0$, then, using (4.4), we can show that

$$P_{\ell\mathrm{cr}}^*(\omega,\epsilon) \sim 4\rho_0^2 U_{\mathrm{c}}^3 \left(\frac{\omega\Delta}{c}\right)^3 \left(\frac{\omega a}{c}\right)^2 \left(\frac{\Delta}{a}\right) \Delta \left\{\epsilon\left(1 - \ln\left(2\frac{\omega^2 a^2}{c^2}\epsilon\right)\right)^2 + \frac{\epsilon\pi^2}{6}\right\} \hat{R}_\ell\left(1, 0, \frac{\omega\Delta}{U_{\mathrm{c}}}\right). \tag{4.6}$$

(*a*) *Case $\omega a/c \ll 1$*

When the radius of the cylinder is small compared with the acoustic wavelength ($\Delta \ll a \ll c/\omega$), for the $n = 0$ mode, the low-wavenumber approximation, $P_\ell^*(k,0,\omega)$, to spectral density is, from (4.2), (3.7) and (3.8), on approximating $F(n,\gamma a)$ by (4.4),

$$P_\ell^*(k,0,\omega) \sim \rho_0^2 U_{\mathrm{c}}^3 \Delta^3 \left(\frac{\Delta}{a}\right)^2 \left\{(ka)^4 f_{1111} \ln^2(|\gamma a|) - 2\,\mathrm{Re}\left\{(ka)^2 g_{011} \ln(\bar{\gamma} a)\right\} + h_0\right\}. \tag{4.7}$$

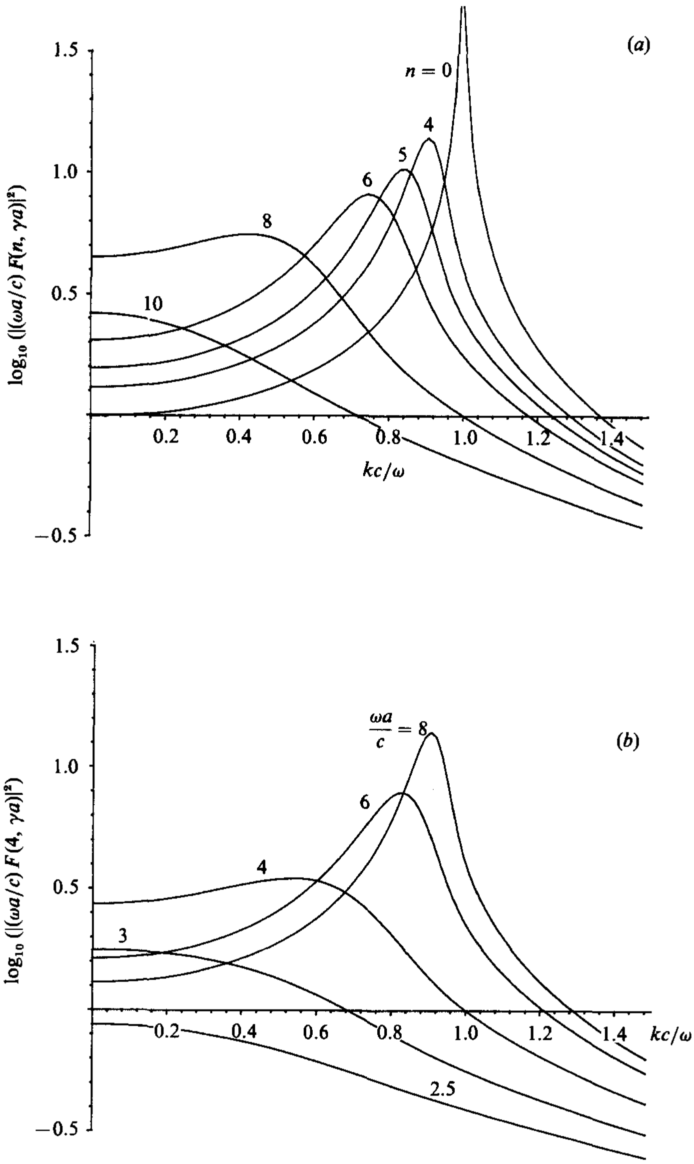


FIGURE 2. (a) Variation of the response function $|(\omega a/c)F(n,\gamma a)|^2$ with the streamwise wavenumber k for different values of n with $\omega a/c = 8$. $|(\omega a/c)F(0,\gamma a)|^2$ has a logarithmic singularity at $k = \omega/c$. (b) Variation of the response function $|(\omega a/c)F(n,\gamma a)|^2$ with k for different values of $\omega a/c$ with $n = 4$.

This can be shown to be in agreement with the leading-order approximation obtained by J. E. Ffowcs Williams (personal communication) and by Chase & Noiseux (1982) (but see remarks following (3.12)).

For other values of n, some knowledge about the structure of the coefficients $f(n, \omega\Delta/U_c)$, etc., is necessary. However for $\omega a/c \ll 1$, the multiplicative factor $|\gamma a F(n,\gamma a)|^2$ is relatively much smaller. Thus, provided this factor principally governs the size of the spectral density for $n \neq 0$, the contribution $P^*_\ell(\omega)$ to the point spectrum $P^*(\omega)$ from the low-wavenumber domain $|k| < a^{-1}$, given by

$$P^*_\ell(\omega) = \int_{|k|<a^{-1}} \mathrm{d}k \left\{ \frac{1}{a} \sum_n P^*_\ell(k, n, \omega) \right\},$$
$$= O\left(\rho_0^2 U_c^3 \Delta \left(\frac{\Delta}{a} \right)^4 h_0 \right). \tag{4.8}$$

(*b*) *Case* $\omega a/c = O(1)$

When the radius of the cylinder is comparable with the acoustic wavelength, the dominant contribution to the point spectrum from the low-wavenumber domain corresponds to moderate values of n. For a given value of n, the variation of the response function $|(\omega a/c) F(n, \gamma a)|^2$ with $\omega a/c$ is shown in figure 2(*b*). For $\omega a/c \geqslant n$, the response function has a peak in the radiative domain, with the peak value increasing and peak width decreasing with increase in $\omega a/c$; for $\omega a/c < n$, the maximum value of the function corresponds to $k = 0$.

We define the line spectrum as

$$P^*(k, \omega) = \frac{1}{a} \sum_n P^*(k, n, \omega). \tag{4.9}$$

In the low-wavenumber domain, $|k| < \Delta^{-1}$,

$$P^*(k, \omega) = \rho_0^2 U_c^3 \left(\frac{\omega \Delta}{c} \right)^2 \frac{\Delta^3}{a} \sum_n \left| \frac{\omega a}{c} F(n, \gamma a) \right|^2 \begin{cases} \hat{R}_\ell \left(\dfrac{kc}{\omega}, \dfrac{nc}{\omega a}, \dfrac{\omega \Delta}{U_c} \right), & |n| < a\Delta^{-1}, \\ \hat{R}_\ell \left(\dfrac{kc}{\omega}, \dfrac{c}{\omega \Delta}, \dfrac{\omega \Delta}{U_c} \right), & |n| > a\Delta^{-1}. \end{cases} \tag{4.10}$$

The line spectrum has a logarithmic singularity at the acoustic streamwise wavenumber. We can estimate $P^*(k, \omega)$ by expanding the factor $\hat{R}_\ell$ for each n about $n = 0$, $\gamma = 0$ and retaining only the leading-order term; that is, for each n replace $\hat{R}_\ell$ in (4.10) by $\hat{R}_\ell(1, 0, \omega\Delta/U_c)$. The corresponding $P^*(k, \omega)$ is shown in figure 3(*a*) for various values of $\omega a/c$ and the expected contribution to the point spectrum from the low-wavenumber region is shown in figure 3(*b*).

(*c*) *Case* $\omega a/c \gg 1$

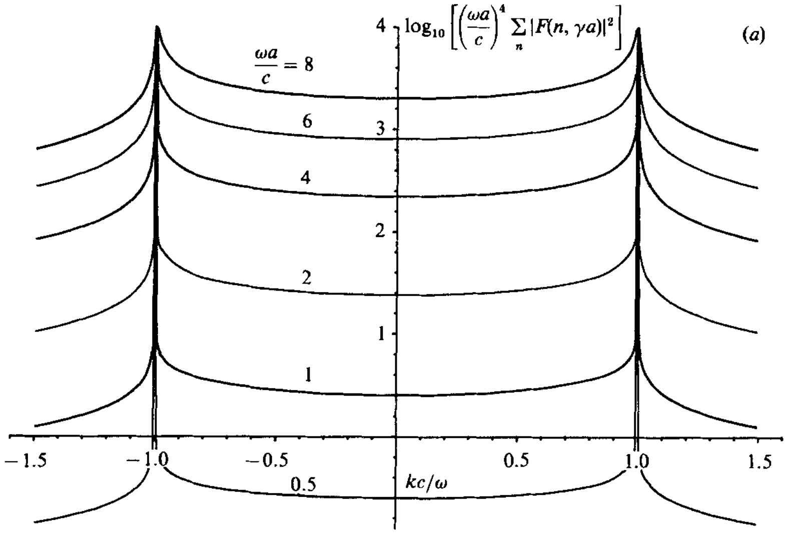
If the radius of the cylinder is large compared with acoustic wavelength, we can approximate the factor $\hat{R}_\ell$ in (4.2) by the corresponding factor for an infinite plane surface with $k_3 = n/a$. Thus the surface pressure spectrum for $\kappa \ll \Delta^{-1}$ is

$$P^*_\ell(k, n, \omega) = \rho_0^2 U_c^3 \Delta^3 \left(\frac{\omega \Delta}{c} \right)^2 \left| \frac{\omega a}{c} F(n, \gamma a) \right|^2 \hat{R}_{\infty\ell} \left(\frac{kc}{\omega}, \frac{nc}{\omega a}, \frac{\omega \Delta}{U_c} \right), \tag{4.11}$$

where $\hat{R}_{\infty\ell}(kc/\omega, k_3 c/\omega, \omega\Delta/U_c)$ is given by (3.17). For each non-zero n, the deterministic factor $|(\omega a/c) F(n, \gamma a)|^2$ has a dominant peak in the vicinity of the circle $\kappa = \omega/c$.

In the case $n = 0$, the behaviour near the acoustic wavenumber is again dominated by the logarithmic singularity of the deterministic factor, and

$$P^*_\ell(k, 0, \omega) \sim \rho_0^2 U_c^3 \Delta^3 \left(\frac{\omega \Delta}{c} \right)^2 \left(\frac{\omega a}{c} \right)^2 \ln^2(|\gamma a|) f_{1111}, \quad \left| k^2 - \frac{\omega^2}{c^2} \right|^{\frac{1}{2}} \ll \frac{1}{a}, \tag{4.12}$$



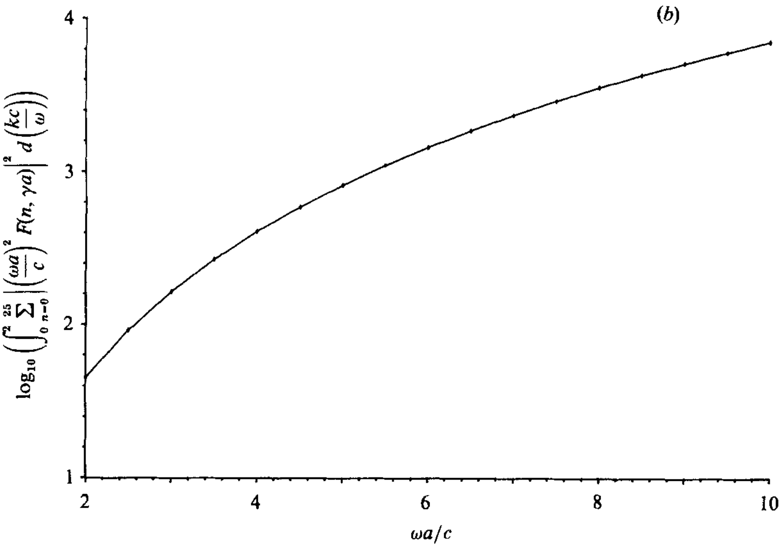


FIGURE 3. (a) Estimated low-wavenumber variation of the line spectrum

$$P^*(k,\omega)\bigg/\left(2\rho_0^2 U_c^3 \Delta^2 \left(\frac{\Delta}{a}\right)^3 \hat{R}_\ell\left(1, 0, \frac{\omega\Delta}{U_c}\right)\right) \approx \left(\frac{\omega a}{c}\right)^4 \sum_n |F(n,\gamma a)|^2$$

with k for various values of $\omega a/c$. The line spectrum has a logarithmic singularity at $k = \pm\omega/c$. (b)

$$P_\ell^*(\omega)\bigg/\left(2\rho_0^2 U_c^3 \Delta \left(\frac{\Delta}{a}\right)^3 \hat{R}_\ell\left(1, 0, \frac{\omega\Delta}{U_c}\right)\right) \approx \left(\frac{\omega a}{c}\right)^4 \int_0^2 \sum_{n=0}^{25} |F(n,\gamma a)|^2 \, d\left(\frac{kc}{\omega}\right)$$

plotted against $\omega a/c$; $P_\ell^*(\omega)$ is the estimated contribution to the point spectrum from the low-wavenumber domain.

on substituting for $\hat{R}_{\infty\ell}(1, 0, \omega\Delta/U_c)$ from (3.17) in (4.11). Away from the singularity we note that

$$\left|\frac{\omega a}{c}F(0,\gamma a)\right|^2 \sim \frac{(\omega/c)^2}{|\gamma|^2} \quad \text{for } \left|k^2-\frac{\omega^2}{c^2}\right|^{\frac{1}{2}} \gg \frac{1}{a}. \tag{4.13}$$

For non-zero values of n, we write $a = n/k_3$ in $|F(n,\gamma a)|^2$ so that

$$-(\gamma a)^2 = (\psi a)^2 = n^2-(\Gamma a)^2 = n^2\left[1-\left(\frac{\Gamma}{k_3}\right)^2\right], \tag{4.14}$$

where Γ is given by (3.4). With $\omega a/c$ large, the peaks correspond to n large and $k_3 = O(\omega/c)$. For moderately large values of n, $|F(n, in(1-(\Gamma/k_3)^2)^{\frac{1}{2}})|^2$ is plotted against $(\Gamma/k_3)^2$ in figure 4. For each n, the peak in $|F|^2$ corresponds to $1 \geqslant (\Gamma/k_3)^2 > 0$; we note that $|\Gamma/k_3| \leqslant 1$ denotes the radiative region $k^2 < \omega^2/c^2$. As n is increased, the position of the peak moves closer to $\Gamma^2 = 0$, while the height and the width of the peak decrease. For large n, using the asymptotic expansions for Hankel functions, we can show that

$$\left|\frac{\omega a}{c}F(n,\gamma a)\right|^2 = n^{-\frac{4}{3}}\left(\frac{\omega a}{c}\right)^2 |\zeta| \left|\frac{\Gamma}{k_3}\right|^{-2} [|\hat{f}(n^{\frac{2}{3}}\zeta)|^2+O(n^{-\frac{2}{3}})], \tag{4.15}$$

where
$$\hat{f}(x) = -\frac{\mathrm{Ai}(x)-\mathrm{iBi}(x)}{\mathrm{Ai}'(x)-\mathrm{iBi}'(x)},$$

$$\tfrac{2}{3}\zeta^{\frac{3}{2}} = \tfrac{1}{2}\ln\left(\frac{1+\Gamma/k_3}{1-\Gamma/k_3}\right)-\Gamma/k_3,$$

and Ai, Bi are the Airy functions.

For $|\Gamma/k_3| \gg n^{-\frac{1}{3}}$ (note that $|\Gamma/k_3|$ may be less than 1), the argument of $\hat{f}(x)$ in (4.15) is large and we may use the asymptotic expansions for Airy functions to show that

$$\left|\frac{\omega a}{c}F(n,\gamma a)\right|^2 \sim \frac{(\omega/c)^2}{|\Gamma|^2}, \quad \left|\kappa^2-\frac{\omega^2}{c^2}\right|^{\frac{1}{2}} \gg k_3 n^{-\frac{1}{3}}. \tag{4.16}$$

Substituting (4.13) and (4.16) into (4.12) and noting that by similar arguments to those above, away from the acoustic wavenumber (A 4) is approximated by (A 6), we have that

$$P_\ell^*(k,n,\omega) = P_{\infty\ell}^*\left(k,\frac{n}{a},\omega\right), \quad \left|\kappa^2-\frac{\omega^2}{c^2}\right|^{\frac{1}{2}} \gg \begin{cases} 1/a, & n=0, \\ k_3 n^{-\frac{1}{3}}, & n \neq 0. \end{cases} \tag{4.17}$$

where $P_{\infty\ell}^*(k, k_3, \omega)$ is the low-wavenumber spectral density for pressure on an infinite plane surface and is given by (3.17a). Thus, except for the vicinity of the total acoustic wavenumber, the spectral density for surface pressure on a large circular cylinder may be approximated by that for an infinite plane surface.

For $|\Gamma/k_3| < 1$, we have

$$\zeta = 2^{-\frac{2}{3}}\left(\frac{\Gamma}{k_3}\right)^2\mu\left(\frac{\Gamma}{k_3}\right); \quad \mu(x) = \left(1+3\left(\frac{x^2}{5}+\frac{x^4}{7}+\dots\right)\right)^{\frac{2}{3}}, \tag{4.18}$$

so that
$$\left|\frac{\omega a}{c}F(n,\gamma a)\right|^2 = 2^{-\frac{2}{3}}n^{-\frac{4}{3}}\left(\frac{\omega a}{c}\right)^2\mu\left(\frac{\Gamma}{k_3}\right)[|\hat{f}(n^{\frac{2}{3}}\zeta)|^2+O(n^{-\frac{2}{3}})]. \tag{4.19}$$

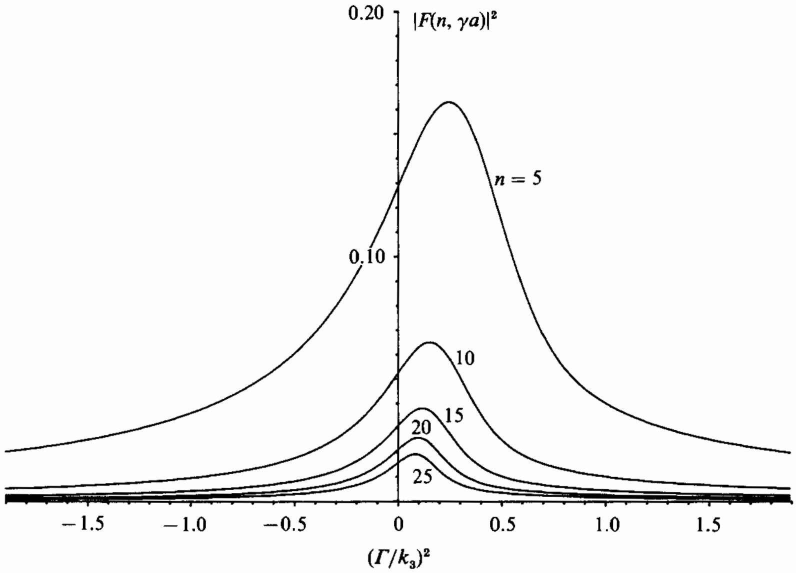


FIGURE 4. Variation of $|F(n,\gamma a)|^2$ with $(\Gamma/k_3)^2$ for large values of n; $\Gamma^2 = \kappa^2-\omega^2/c^2$, $k_3 = n/a$ and $(\gamma a)^2 = a^2(k^2-\omega^2/c^2) = n^2((\Gamma/k_3)^2-1)$.

The maximum value of the right-hand side of (4.19) approximately corresponds to the value of $X > 0$, for which $(\mathrm{d}/\mathrm{d}X)\,|\hat{f}(X)|^2 = 0$ and is given by

$$\left(\frac{\Gamma}{k_3}\right)^2 = d_0^2\, n^{-\frac{2}{3}}, \quad d_0 = 0.837, \tag{4.20}$$

that is, where $\kappa^2 = (\omega/c)^2 + d_0^2\, n^{-\frac{2}{3}} k_3^2$. In the neighbourhood of the maximum

$$\left|\frac{\omega a}{c} F(n,\gamma a)\right|^2 \sim d_0^{-2}\, n^{-\frac{4}{3}} \left(\frac{\omega a}{c}\right)^2 \left(1 - d_{02}\left(\frac{\Gamma_1}{k_3}\right)^4\right), \tag{4.21}$$

where $\Gamma_1^2 = \Gamma^2 - d_0^2\, n^{-\frac{2}{3}} k_3^2$, and $d_{02} = 2.288$.

Substituting (4.19) into (4.11), we have that the low-wavenumber pressure spectrum for $\omega a/c \gg 1$, $n \gg 1$ is

$$P_\ell^*(k,n,\omega) \approx 2^{-\frac{2}{3}} n^{-\frac{4}{3}} \rho_0^2\, U_c^3\, \Delta^3 \left(\frac{\omega\Delta}{c}\right)^2 \left(\frac{\omega a}{c}\right)^2 \mu\left(\frac{\Gamma}{k_3}\right) |\hat{f}(n^{\frac{2}{3}}\zeta)|^2 \hat{R}_{\infty\ell}\left(\frac{kc}{\omega}, \frac{nc}{\omega a}, \frac{\omega\Delta}{U_c}\right). \tag{4.22}$$

The peak value $P_{\ell\max}^*$ is approximately give by

$$P_{\ell\max}^* = n^{-\frac{4}{3}} \rho_0^2\, U_c^3\, \Delta^3 \left(\frac{\omega\Delta}{c}\right)^2 \left(\frac{\omega a}{c}\right)^2 d_0\, \hat{R}_{\infty\ell}\left(\sin\chi,\, \cos\chi,\, \frac{\omega\Delta}{U_c}\right), \tag{4.23}$$

where $\tan\chi = ak/n$, so that $P_{\ell\max}^* = O(M^4(\omega a/U_c)^2 n^{-\frac{4}{3}})$ or equivalently $O(M^4(\omega^2/U_c^2)\, a^{\frac{2}{3}} k_3^{-\frac{4}{3}})$. Although the peak is finite, its value is large for large values of the cylinder radius. The peak width is $O(n^{\frac{2}{3}}/a)$, or equivalently $O(k_3^{\frac{2}{3}}\, a^{-\frac{1}{3}})$.

4.1. *Effect of curvature on an almost plane cylinder*

We write
$$\lambda = \frac{\omega a}{c} \tag{4.24}$$
and note that, in view of the above considerations for $\lambda \gg 1$, we may approximate the factor $|\lambda F(n, \gamma a)|^2$ by
$$|\lambda F(n,\gamma a)|^2 \approx \frac{(\omega/c)^2}{|\kappa^2(1-\beta^2)-\omega^2/c^2|+\kappa^2\beta^2}, \tag{4.25}$$
where $\beta(\chi) = d_0|\cos\chi|^{\frac{2}{3}}/\lambda^{\frac{1}{3}}$, and χ is as in (4.23) and d_0 is given by (4.20). The approximate function has a maximum at $\kappa = \pm(\omega/c)(1-\beta^2)^{-\frac{1}{2}}$. Except for $\chi = \frac{1}{2}\pi$, the maximum value is finite and equal to $1/\beta^2$ and the width of the peak is of $O(\beta)$. Away from the peak, for $|\kappa^2-\omega^2/c^2| \gg (\omega^2/c^2)\lambda^{-\frac{2}{3}}$, the approximation reflects the behaviour of the corresponding deterministic factor for the infinite plane surface (this being exactly so for $\kappa^2-\omega^2/c^2 > (\omega^2/c^2)\beta^2(1-\beta^2)^{-1}$). For $\chi = \frac{1}{2}\pi$, the approximation is still singular but, except for the case of axisymmetric flow, we show below that the singularity is integrable. The axisymmetric case needs to be considered separately and is dealt with in Appendix B.

If we let $n/a \to k_3$, then, from (4.11), the surface pressure spectral density is approximately given by
$$P_\ell^*(k,k_3,\omega) \approx \rho_0^2 U_c^3\left(\frac{\omega\Delta}{c}\right)^4 \Delta \frac{\hat{R}_{\infty\ell}\left(\dfrac{\kappa c}{\omega}\sin\chi, \dfrac{\kappa c}{\omega}\cos\chi, \dfrac{\omega\Delta}{U_c}\right)}{\left|\kappa^2(1-\beta^2)-\dfrac{\omega^2}{c^2}\right|+\kappa^2\beta^2}. \tag{4.26}$$

This approximate form differs from that given recently by Howe (1987) in that instead of the factor $(|\kappa^2(1-\beta^2)-\omega^2/c^2|+\kappa^2\beta^2)^{-1}$ he chooses to use $(|\kappa^2-\omega^2/c^2| + (\omega^2/c^2)\beta_0^2)^{-1}$ where $\beta_0 = (0.92/0.837)\beta$. It can be shown that for $k_3 > 0$ the latter choice underestimates the height of the peak by the $O(1)$ factor β^2/β_0^2; the difference arises since Howe approximates the position of the peak to lie at the acoustic wavenumber, $|\kappa| = \omega/c$.

The contribution to the point pressure spectrum from the low-wavenumber domain, $\kappa \ll \Delta^{-1}$, is approximately given by
$$P_\ell^*(\omega) = \int_{-\pi}^{\pi}\int_0^{2(\omega/c)} P_\ell^*(\kappa\cos\chi, \kappa\sin\chi, \omega)\,\kappa\,\mathrm{d}\kappa\,\mathrm{d}\chi, \tag{4.27}$$
where we have replaced the summation with respect to n by the appropriate integral with respect to the wavenumber k_3.

We can estimate $P_\ell^*(\omega)$ if we approximate the factor
$$\hat{R}_{\infty\ell}\left(\frac{\kappa c}{\omega}\sin\chi, \frac{\kappa c}{\omega}\cos\chi, \frac{\omega\Delta}{U_c}\right)$$
in (4.26) by its value at the total acoustic wavenumber, $|\kappa| = \omega/c$; that is, if we approximately set it to $\hat{R}_{\infty\ell}(\sin\chi, \cos\chi, \omega\Delta/U_c)$. For then
$$P_\ell^*(\omega) = \rho_0^2 U_c^3\left(\frac{\omega\Delta}{c}\right)^4 \Delta \int_{-\pi}^{\pi}\int_0^{2(\omega/c)} \frac{\hat{R}_{\infty\ell}\left(\sin\chi, \cos\chi, \dfrac{\omega\Delta}{U_c}\right)}{\left|\kappa^2(1-\beta^2)-\dfrac{\omega^2}{c^2}\right|+\kappa^2\beta^2}\,\kappa\,\mathrm{d}\kappa\,\mathrm{d}\chi. \tag{4.28}$$

The integral can be evaluated approximately to give

$$P_\ell^*(\omega) = 2\pi\rho_0^2 U_c^3 \left(\frac{\omega\Delta}{c}\right)^4 \Delta\, B(\omega_1)\left[\ln\left(\frac{2(4\lambda)^{\frac{2}{3}}}{d_0^2}\right) + O\left(\lambda^{-\frac{2}{3}}, \left(\frac{\omega\Delta}{c}\right)^2\right)\right], \qquad (4.29a)$$

where $\omega_1 = \omega\Delta/U_c$, and

$$B(\xi) = \tfrac{1}{8}[f_{iijj}(\xi) + f_{ijij}(\xi)] \quad (i,j = 1,3),$$

the summation convention being implied and f_{ijlm} being the functions of ω_1 which appear in the definition of $\hat{R}_{\infty\ell}$. In terms of the Mach number M,

$$P_\ell^*(\omega) = 2\pi M^4 \rho_0^2 U_c^3 \Delta\, B_1(\omega_1)\left[\ln\left(2d_0^{-2}\left(\frac{4M\omega_1 a}{\Delta}\right)^{\frac{2}{3}}\right) + O\left(\left(\frac{M\omega_1 a}{\Delta}\right)^{-\frac{2}{3}}, M^2\right)\right] \qquad (4.29b)$$

where $B_1(\omega_1) = \omega_1^4 B(\omega_1)$. That is,

$$P_\ell^*(\omega) = O\left(\rho_0^2 U_c^3 \Delta M^4 \ln\left(\frac{Ma}{\Delta}\right)^{\frac{2}{3}}\right) \qquad (4.29c)$$

so that the logarithmic factor is somewhat greater than in the corresponding expression in the axisymmetric case considered in Appendix B.

5. Discussion

We have shown that apart from a logarithmic, integrable, singularity at $k = \omega/c$, $n/a = 0$, the cross-spectral density of pressure on the surface of a cylinder due to statistically stationary turbulent flow is finite, even though the flow is regarded as being inviscid. For each non-zero n, $n \leqslant \omega a/c$, it attains a peak value at a total wavenumber κ_m where

$$\frac{\omega^2}{c^2} + \frac{n^2}{a^2} > \kappa_m^2 > \frac{\omega^2}{c^2};$$

for n such that $a\Delta^{-1} > |n| > \omega a/c$, it has a maximum value at $k = 0$. For $\omega a/c \gg 1$, the peak value has a coefficient $O[(\omega a/c)^2 n^{-\frac{4}{3}}]$ and a width $O[(\omega/c)(\omega a/c)^{-\frac{1}{3}}]$; away from the peak, the spectrum coincides with that for an infinite plane surface. The peak value is large but finite.

The expression (4.2) for the low-wavenumber pressure spectrum contains a number of unknown functions of frequency which need to be determined experimentally. Some of these can be determined by considering the radiated sound, as suggested by Bergeron (1973). If we write $P(\xi, r_1, \eta, \zeta)$ for the pressure correlation at a fixed distance $r_1 > a + \Delta$,

$$P(x-\tilde{x}, r_1, \theta-\tilde{\theta}, t-\tilde{t}) = \langle p(x, r_1, \theta, t)\, p(\tilde{x}, r_1, \tilde{\theta}, \tilde{t})\rangle, \qquad (5.1)$$

then, from (2.10), we can show that for $\Delta/a < 1$, the power spectrum, $P^*(r_1 k, n, \omega)$ is approximately given by

$$P^*(r_1, k, n, \omega) = \frac{|K_n(\gamma r_1)|^2}{|K_n(\gamma a)|^2} P^*(k, n, \omega), \qquad (5.2)$$

where $P^*(k, n, \omega)$ is the power spectrum for the surface pressure and is given by (3.5), and γ is given by (2.11). For $\Delta \ll r_1 - a \ll a$, we have, approximately,

$$P^*(r_1, k, n, \omega) = \begin{cases} P_\ell^*(k, n, \omega), & \gamma^2 \leqslant a^{-2} \\ 0, & \gamma^2 > \Delta^{-2}. \end{cases} \qquad (5.3)$$

where $P_\ell^*(k, n, \omega)$ is given by (4.2).

The intensity of radiated sound is given by

$$I = \frac{1}{\rho_0 c}\langle (p-p_0)^2\rangle. \tag{5.4}$$

For a small cylinder, $\Delta \ll a \ll c/\omega$, we have from (4.8)

$$I = O\left(\rho_0 \frac{U_c^4}{c}\left(\frac{\Delta}{a}\right)^4 \int_{-\infty}^{\infty} h_0(\omega_1)\,\mathrm{d}\omega_1\right). \tag{5.5}$$

For a cylinder of moderate size, $\omega a/c = O(1)$, we have from (4.10) that $I = O(\rho_0 U_c^8/c^5)$.

For a large cylinder, $\omega a/c \gg 1$, we have from (4.29) that

$$I = \rho_0 \frac{U_c^8}{c^5}\left[A_0 \ln\left[2\left(\frac{4Ma}{\Delta}\right)^{\frac{2}{3}}\right] + O(1)\right], \tag{5.6}$$

with
$$A_0 = 2\pi \int_{-\infty}^{\infty} B_1(\omega_1)\,\mathrm{d}\omega_1,$$

$B_1(\omega)$ as in (4.29*b*) and M the flow Mach number. In the case of axisymmetric flow, with $\omega a/c \gg 1$, we have from (B 8) that

$$I = \rho_0 \frac{U_c^8}{c^5}\left[A_1 \ln\left(\frac{Ma}{\Delta}\right) + O(1)\right], \tag{5.7}$$

where
$$A_1 = 4\int_{-\infty}^{\infty} \omega_1^4 f_{1111}(\omega_1)\,\mathrm{d}\omega_1$$

and $f_{1111}(\omega_1)$ is defined in (A 3).

It is interesting to note that for a plane surface of large but finite size of typical dimension L, Bergeron showed that the intensity of radiated sound was $O[\rho_0(U_c^8/c^5)\ln(ML/\Delta)]$. With L replaced by $L_1 = 4(2a^2\Delta)^{\frac{1}{3}}M^{-\frac{1}{3}}$ in this expression, the logarithmic factor is the same as in the expression (5.6); the corresponding result for the axisymmetric case requires L to be replaced by a. The difference in the two cases reflects the dominance of different contributions to the intensity at a point close to the surface. In the non-axisymmetric case the dominant contribution arises from creeping rays emanating from turbulent sources within distance L_1 (measured along the surface of the cylinder) from the point, L_1^{-1} being the decay rate of creeping rays on a cylinder. In the axisymmetric case, we expect the dominant contribution to come from turbulent sources in line of sight of the point.

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Appendix A. Coefficients associated with the source functions

With θ_1, θ_3 and Γ defined by (3.3) and (3.4), we introduce

$$(\Gamma\Theta)^2 = \gamma^2 + \left(\frac{n}{r}\right)^2; \quad (\Gamma\tilde{\Theta})^2 = \gamma^2 + \left(\frac{n}{\tilde{r}}\right)^2 \tag{A 1}$$

and

$$\phi_1 = \tilde{\phi}_1 = 1, \quad \phi_3 = \frac{a}{r}, \quad \tilde{\phi}_3 = \frac{a}{\tilde{r}}. \tag{A 2}$$

Then the expressions f, g, h in the factor $\hat{R}(k, n, \omega)$ of equation (3.7) are given by

$$\left.\begin{aligned}
&f_{ijlm} = [\phi_i \phi_j \tilde{\phi}_l \tilde{\phi}_m J^*_{ijlm}]_F, && f_0 = [\Theta^2 \tilde{\Theta}^2 J^*_{rrrr}]_F,\\
&f_{23lm} = \mathrm{Im}\,[\phi_l \phi_m \phi_3^2 J^*_{r\theta lm}]_F, && f_{2323} = [\phi_3^2 \tilde{\phi}_3^2 J^*_{r\theta r\theta}]_F,\\
&f_{22lm} = \mathrm{Re}[\Theta^2 \tilde{\phi}_l \tilde{\phi}_m J^*_{rrlm}]_F, && f_{3222} = \mathrm{Im}\,[-\tilde{\Theta}^2 J^*_{\theta rrr}]_F,\\
&g_0 = [\tilde{\Theta}^2(\phi_3 J^*_{\theta\theta rr} - \tilde{\phi}_3 J^*_{rrrr})]_G, && g_{j222} = [\mathrm{i}\tilde{\Theta}^2 \phi_j J^*_{jrrr}]_G,\\
&g_{j2lm} = [\mathrm{i}\phi_j \tilde{\phi}_l \tilde{\phi}_m J^*_{jrlm}]_G, && g_{i223} = [\phi_i \tilde{\phi}_3^2 J^*_{irr\theta}]_G,\\
&g_{0lm} = [\tilde{\phi}_l \tilde{\phi}_m(\tilde{\phi}_3 J^*_{\theta\theta lm} - \phi_3 J^*_{rrlm})]_G, && g_1 = [\mathrm{i}\tilde{\phi}_3^2 \phi_3(J^*_{\theta\theta r\theta} - J^*_{rrr\theta})]_G,\\
&h_{i22l} = [\phi_i \tilde{\phi}_l J^*_{irrl}]_H, && h_{0j} = -\mathrm{Im}\,[\tilde{\phi}_3 \phi_j(J^*_{jr\theta\theta} - J^*_{jrrr})]_H,\\
&h_0 = [\phi_3 \tilde{\phi}_3(J^*_{\theta\theta\theta\theta} - J^*_{rr\theta\theta} - J^*_{\theta\theta rr} + J^*_{rrrr})]_H,
\end{aligned}\right\} \tag{A 3}$$

where the indices take the values, $i, j, l, m = 1, 3$; f, g, h are non-dimensional functions of (k, n, ω) and the notation on the right-hand side of the expressions is defined as

$$\left.\begin{aligned}
[f]_F &= (\rho_0^2 U_c^3 \Delta^5)^{-1} \iint_a^\infty \frac{K_n(\gamma r)\bar{K}_n(\gamma\tilde{r})}{|aK_n(\gamma a)|^2} f r\tilde{r}\,\mathrm{d}r\,\mathrm{d}\tilde{r},\\
[f]_G &= (\rho_0^2 U_c^3 \Delta^5)^{-1} \iint_a^\infty \frac{K'_n(\gamma r)\bar{K}_n(\gamma\tilde{r})}{a^2 K'_n(\gamma a)\bar{K}_n(\gamma a)} f r\tilde{r}\,\mathrm{d}r\,\mathrm{d}\tilde{r},\\
[f]_H &= (\rho_0^2 U_c^3 \Delta^5)^{-1} \iint_a^\infty \frac{K'_n(\gamma r)\bar{K}'_n(\gamma\tilde{r})}{a^2 K'_n(\gamma a)\bar{K}'_n(\gamma a)} f r\tilde{r}\,\mathrm{d}r\,\mathrm{d}\tilde{r}.
\end{aligned}\right\} \tag{A 4}$$

In the low-Mach-number, low-wavenumber approximation, we expand the integrand in (A 4) about $M = 0$, $k = 0$ to obtain

$$[f]_F \approx (\rho_0^2 U_c^3 \Delta^5)^{-1} \iint_a^\infty \frac{a^{2n}}{(r\tilde{r})^n} f \frac{r\tilde{r}}{a^2}\,\mathrm{d}r\,\mathrm{d}\tilde{r},$$

with similar expressions for $[f]_G$ and $[f]_H$.

In the appropriate limiting case of the plane surface (see §3.1), we set

$$\phi_1 = \tilde{\phi}_1 = \phi_3 = \tilde{\phi}_3 = 1 \tag{A 5}$$

in the above and note that in this case

$$[f]_F = [f]_G = [f]_H = (\rho_0^2 U_c^3 \Delta^5)^{-1} \iint_0^\infty \mathrm{e}^{-(\Gamma y + \tilde{\Gamma}\tilde{y})} f\,\mathrm{d}y\,\mathrm{d}\tilde{y}. \tag{A 6}$$

Then the functions f, g and h which appear in the expression (3.15) for $\hat{R}_\infty(k, k_3, \omega)$ are defined by (A 3), (A 5) and (A 6) with subscript θ replaced by z in the source functions J.

Appendix B: Axisymmetric case for $\omega a/c \gg 1$

For $\kappa \ll \Delta^{-1}$ and $\lambda = \omega a/c \gg 1$, the spectral density for the surface pressure in the case of axisymmetric flow is, from (4.11),

$$P_{\ell}^{*}(k,0,\omega) = \lambda^2 \rho_0^2 U_c^3 \left(\frac{\omega\Delta}{c}\right)^2 \Delta^3 \frac{|K_0(\gamma a)|^2}{|\gamma a K_0'(\gamma a)|^2} \tilde{R}_{\infty\ell}\left(\frac{kc}{\omega}, 0, \frac{\omega\Delta}{U_c}\right), \tag{B 1}$$

where from (3.17)

$$\tilde{R}_{\infty\ell}(\xi,0,\eta) = \xi^4 f_{1111}(\eta) - 2\xi^2[(\xi^2-1)f_{2211}(\eta) - 2|\xi^2-1|\,h_{1221}(\eta)] + (\xi^2-1)^2 f_0(\eta) - 4\xi\,\mathrm{Re}\{(\xi^2-1)^{\frac{1}{2}}[(\xi^2-1)\,g_{1221}(\eta) + \xi^2 g_{1211}(\eta)]\}. \tag{B 2}$$

The contribution to the point spectrum from the low-wavenumber domain is given by

$$P_{\ell}^{*}(\omega) = \frac{\omega}{c}\int_{-2(\omega/c)}^{2(\omega/c)} P_{\ell}^{*}(k,0,\omega)\,\mathrm{d}k. \tag{B 3}$$

If we approximate the factor $\tilde{R}_{\infty\ell}(kc/\omega, 0, \omega\Delta/U_c)$ in (B 1) by $\tilde{R}_{\infty\ell}(1, 0, \omega\Delta/U_c)$ for all $|k| < 2(\omega/c)$, then we can estimate $P_{\ell}^{*}(\omega)$ by

$$P_{\ell}^{*}(\omega) \sim 2\lambda^2 \rho_0^2 U_c^3 \left(\frac{\omega\Delta}{c}\right)^4 \Delta f_{1111} \int_0^\infty \frac{|K_0(\lambda\gamma_1)|^2}{|\lambda\gamma_1 K_0'(\lambda\gamma_1)|^2}\,\mathrm{d}\xi, \tag{B 4}$$

where $\gamma_1 = (\xi^2-1)^{\frac{1}{2}}$ and the upper limit of the integral is set to infinity since the integrand decays as $1/\xi^2$ for large ξ.

Now, for small values of $|\lambda\gamma_1|$,

$$\frac{|K_0(\lambda\gamma_1)|^2}{|\lambda\gamma_1 K_0'(\lambda\gamma_1)|^2} \sim \ln^2(|\lambda\gamma_1|), \quad \lambda\gamma_1 \ll 1 \tag{B 5}$$

and for large values of $|\lambda\gamma_1|$,

$$\frac{|K_0(\lambda\gamma_1)|^2}{|\lambda\gamma_1 K_0'(\lambda\gamma_1)|^2} \sim \frac{1}{|\lambda\gamma_1|^2}, \quad |\lambda\gamma_1| \gg 1, \tag{B 6}$$

so that we can estimate the integral in (B 4) by

$$\int_0^\infty \frac{|K_0(\lambda\gamma_1)|^2}{|\lambda\gamma_1 K_0'(\lambda\gamma_1)|^2}\,\mathrm{d}\xi \sim \int_0^{1-\epsilon_1} \frac{\mathrm{d}\xi}{\lambda^2(1-\xi^2)} + \int_{1-\epsilon_1}^{1+\epsilon_2} \ln^2|\lambda\gamma_1|\,\mathrm{d}\xi + \int_{1+\epsilon_2}^\infty \frac{\mathrm{d}\xi}{\lambda^2(\xi^2-1)}, \tag{B 7}$$

where, $\epsilon_1 \approx \epsilon_2 \approx a_0\lambda^{-2}$, a_0 being $O(1)$. Hence, on evaluating the integrals in (B 7), the point spectrum $P_{\ell}^{*}(\omega)$ in the axisymmetric case is given by

$$\begin{aligned} P_{\ell}^{*}(\omega) &\sim 4\rho_0^2 U_c^3 \left(\frac{\omega\Delta}{c}\right)^4 \Delta f_{1111} \ln\left(\frac{\omega a}{c}\right) \\ &= 4M^4 \rho_0^2 U_c^3 \left(\frac{\omega\Delta}{U_c}\right)^4 \Delta f_{1111} \ln\left(M\frac{\omega_1 a}{\Delta}\right), \end{aligned} \tag{B 8}$$

where M is the Mach number, $M = U_c/c$ and $\omega_1 = \omega\Delta/U_c$.

REFERENCES

BERGERON, R. F. 1973 Aerodynamic sound and the low-wavenumber wall-pressure spectrum of nearly incompressible boundary layer turbulence. *J. Acoust. Soc. Am.* **54**, 123.

CHASE, D. M. 1980 Modelling the wave vector–frequency spectrum of turbulent boundary layer wall pressure. *J. Sound Vib.* **70**, 29.

CHASE, D. M. & NOISEUX, C. F. 1982 Turbulent wall pressure at low wavenumbers: Relation to nonlinear sources in planar and cylindrical flow. *J. Acoust. Soc. Am.* **72**, 975.

CROW, S. C. 1967 Visco-elastic character of fine-grained isotropic turbulence. *Phys. Fluids* **10**, 1587.

CROW, S. C. 1970 Aerodynamic sound emission as a singular perturbation problem. *Stud. Appl. Maths* **49**, 21.

DOWLING, A. P. 1983 The low wavenmber wall pressure spectrum on a flexible surface. *J. Sound Vib.* **88**, 11.

DOWLING, A. P., FFOWCS WILLIAMS, J. E. & GOLDSTEIN, M. E. 1978 Sound production in a moving stream. *Phil. Trans. R. Soc. Lond.* A **288**, 321.

FFOWCS WILLIAMS, J. E. 1965 Surface-pressure fluctuations induced by boundary-layer flow at finite Mach numbers. *J. Fluid Mech.* **22**, 507.

FFOWCS WILLIAMS, J. E. 1982 Boundary-layer pressures and the Corcos model: a development to incorporate low-wavenumber constraints. *J. Fluid Mech.* **125**, 9.

HOWE, M. S. 1979 The role of surface shear stress fluctuations in the generation of boundary layer noise. *J. Sound Vib.* **65**, 159.

HOWE, M. S. 1987 The singularity at the acoustic wavenumber of the turbulent boundary layer wall pressure spectrum. *Proc. R. Soc. Lond.* A **412**, 389.

JONES, D. S. 1979 *Methods in Electromagnetic Wave Propagation*. Clarendon.

LIGHTHILL, M. J. 1952 On sound generated aerodynamically. I. General Theory. *Proc. R. Soc. Lond.* A **211**, 564.